Int. J. Heat Mass Transfer. Vol. 33, No. 1, pp. 183–201, 1990
Printed in Great Britain

0017-9310/90 $3.00+0.00
Pergamon Press plc

An algebraic method for heat and mass transfer problems

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(*Received* 23 *March* 1988)

Abstract—A new approximate analytical method for studying mass and heat transfer problems is presented which makes it possible to obtain simple algebraic equations for diffusion (thermal) flows and mean Sherwood numbers. The problems are considered concerning convective mass transfer to a rotating disk, longitudinally streamlined plate, a spherical droplet, and a solid particle in the course of volumetric reaction the rate of which is arbitrarily dependent on concentration. Non-stationary mass exchange between a wall and a motionless medium with simultaneous volumetric reaction of any order is studied. Heat and mass transfer in a turbulent flow in the tube inlet section and other analogous problems are analysed.

1. INTRODUCTION

1.1. *Introductory remarks*

IT IS WELL known that the solution of linear ordinary arbitrary order differential equations with constant coefficients by the substitution of $y = e^{\lambda x}$ (where y is the dependent and x independent variables) is reduced to the solution of algebraic equations of the same order to determine the exponent λ.

Such an unambiguous correspondence between the mentioned types of differential and algebraic equations suggested the possibility of trying to use similar ideas for an approximate analysis of much more complicated non-linear differential equations with variable coefficients.

The present paper proposes a very simple and effective approximate analytical method for studying linear and non-linear differential equations with variable coefficients (as well as equations with partial derivatives) which is based on the solution of auxiliary algebraic equations with subsequent use of the procedure of asymptotic correction. An elementary technique is described which allows the above-mentioned auxiliary algebraic equations for diffusion flows to be derived mentally.

The suggested method for obtaining approximate relations rests on a well-balanced logical foundation (there exists a relationship between the algebraic method and the combination of the Galerkin method with that of asymptotic correction) and leads, in a number of cases, to accurate results.

In view of simple and formalized principles of actions, the algebraic method is essentially a sort of 'approximate operational method'. It is well illustrated by the solution of different problems which are of interest for the theory of mass and heat transfer, chemical technology and physicochemical hydrodynamics.

1.2. *The method of asymptotic correction*

First, a modified version of the method of asymptotic correction will be described which will be further employed to present the algebraic method.

Let the following expression be obtained for the unknown value S:

$$S = \varphi(p) \qquad (1)$$

which correctly reflects the qualitative behaviour of S as a function of the change in the dominant parameter p. The function φ is taken to be monotonous. Let the main terms of the asymptotic expansion of the approximate expression (1) in the limiting cases $p \to 0$ and $p \to \infty$ have the form

$$p \to 0,\ S \to S_0^*; \quad p \to \infty,\ S \to S_\infty^* \qquad (2)$$

where

$$S_0^* = S_0^*(p), \quad S_\infty^* = S_\infty^*(p). \qquad (3)$$

Here and after it is assumed that $S_\infty^*/S_0^* \neq \text{const.}$

If similar exact asymptotics of the initial problem solution are known

$$p \to 0,\ S \to S_0; \quad p \to \infty,\ S \to S_\infty \qquad (4)$$

then the approximate formula (1) can be improved by the following simple means. Taking into account that, according to equations (3), the asymptotics of the approximate expressions of S_0^* and S_∞^* depend on the parameter p, expression (1) can be given in the following form:

$$\frac{S}{S_0^*} = \Phi\left(\frac{S_\infty^*}{S_0^*}\right). \qquad (5)$$

This equation no longer depends explicitly on p (the substitution of the functions $S = \varphi(p)$, $S_0^* = S_0^*(p)$, $S_\infty^* = S_\infty^*(p)$ from equations (1) and (3) reduces equation (5) to an identity).

NOMENCLATURE

a	characteristic length scale (tube, radius, radius of droplets, bubbles and solid spherical particles)
C	concentration in flow
C_s	concentration on the surface (of a disk, tube, plate, particle, droplet, bubble)
C_∞	unperturbed concentration in a free stream (at the tube inlet; at a distance from a particle, droplet, bubble, disk)
c	dimensionless concentration, $(C_\infty - C)/(C_\infty - C_s)$; in problems with volumetric reaction it should be assumed that $C_\infty = 0$, which corresponds to $c = C/C_\infty$
$\bar{c}$	image of dimensionless concentration, $p\int_0^\infty e^{-p\tau} c \, d\tau$
D	coefficient of diffusion
F	function determining the kinetics of volumetric reaction, $F(C)$; for the Nth-order reaction $F = C^N$
$f(c)$	$F(C)/F(C_s)$; for the Nth-order reaction $f = c^N$
$\langle f \rangle$	$\int_0^1 f(c)\,dc$; for the Nth-order reaction $\langle f \rangle = 1/(N+1)$
K	constant of volumetric reaction rate (KF is the rate of volumetric reaction)
k	dimensionless constant of volumetric reaction rate, $a^2KF(C_s)/DC_s$; for the Nth-order reaction $k = a^2KC_s^{N-1}/D$
Pe	Peclet number, aU_∞/D
r	dimensionless radial coordinate; $r = 1$ corresponds to the surface of a spherical droplet, bubble and solid particle
r, θ	spherical system of coordinates with origin fixed at the centre of a spherical droplet, bubble and a solid particle
Sh	mean Sherwood number, $-\frac{1}{2}\int_0^\pi \sin\theta(\partial c/\partial r)_{r=1}\,d\theta$
t	time
U_∞	unperturbed flow velocity (at a distance from a plate, droplet, bubble, particle)
X	distance (to the surface of a disk, plate, tube, wall)
x	X/a.

Greek symbols

β	ratio of dynamic viscosities of droplet and surrounding liquid ($\beta = 0$ corresponds to a gas bubble; $\beta = \infty$, to a solid particle)
$\Gamma(\alpha)$	gamma-function, $\gamma(\alpha, \infty)$
$\Gamma(\alpha, x)$	$\Gamma(\alpha) - \gamma(\alpha, x)$
$\gamma(\alpha, x)$	incomplete gamma-function, $\int_0^x \zeta^{\alpha-1} e^{-\zeta}\,d\zeta$
δ	boundary layer thickness
ν	kinematic viscosity of fluid
τ	dimensionless time, Dt/a^2.

Now, with asymptotics (3) of the appropriate formula (1) being substituted in expression (5) by the corresponding asymptotics of an exact solution of the initial problem (4), it is possible to obtain the following formula:

$$\frac{S}{S_0} = \Phi\left(\frac{S_\infty}{S_0}\right) \tag{6}$$

which, besides being a correct qualitative description of the quantity S, usually provides an accurate result in the limiting cases $p \to 0$ and $p \to \infty$ (in contrast to equation (1)). This is the essence of the method of asymptotic correction.

Specifically, if asymptotics (3) have the form

$$S_0^* = Ap^\alpha, \quad S_\infty^* = Bp^\beta \quad (\alpha \neq \beta) \tag{3a}$$

then the sought relation (6) can be presented as

$$\frac{S}{S_0} = \frac{1}{A}\left(\frac{A}{B}\frac{S_\infty}{S_0}\right)^{\alpha/(\alpha-\beta)} \varphi\left(\left(\frac{A}{B}\frac{S_\infty}{S_0}\right)^{1/(\beta-\alpha)}\right). \tag{6a}$$

It is usually assumed that the structure of accurate asymptotics S_0 and S_∞ is analogous to that of equation (3a): the exponents α and β are the same, but the coefficients A and B are different.

Figure 1 schematically illustrates the application of the method of asymptotic correction when $\alpha = 0$ and $\beta > 0$.

Similarly, instead of the explicit initial relation (1), the implicit approximate relation $G(S, p) = 0$ can be considered.

To illustrate the derivation of the sought function Φ, equation (6), the following quadratic equation will be considered (an example of implicit dependence)

$$a_2S^2 + pS + a_0 = 0. \tag{7}$$

At small p, equation (7) yields the 'truncated' equation

$$a_2S_0^{*2} + a_0 = 0 \quad (p \to 0) \tag{8}$$

while at large p it gives for the growing asymptotic

$$a_2S_\infty^* + p = 0 \quad (p \to \infty). \tag{9}$$

Now, elimination of the parameters a_0 and p from equation (7) with the aid of equations (8) and (9) yields

$$S^2 - S_\infty S - S_0^2 = 0 \tag{10}$$

where the asterisk at the asymptotics of the quantity

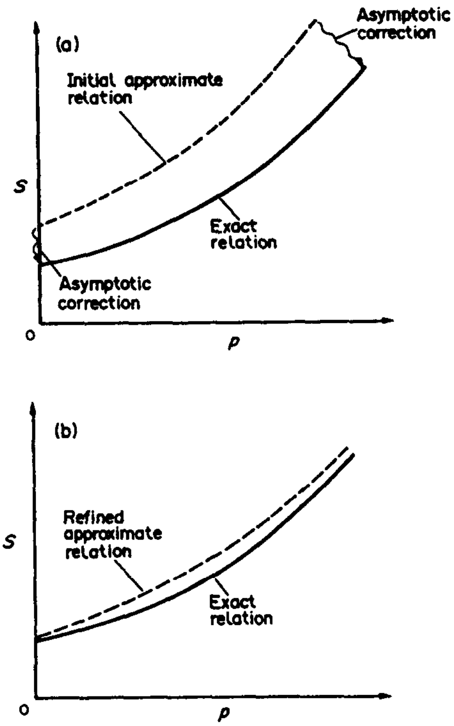

FIG. 1. Scheme of the asymptotic correction method operation for refining approximate relations.

S has been already omitted for the ease of representation. Equation (10) can be easily presented in the form of equation (6).

Note that a rather comprehensive treatment of the method of asymptotic correction for the functions which depend on two parameters is made in refs. [1, 2]. Also given there are a great number of specific examples showing the use of this method in the problems of convective mass and heat transfer.

1.3. *Guiding considerations*

Before beginning a discussion of the algebraic method, it will be useful first to consider simple arguments which provided the starting point for its formulation.

It is well known that the solution of linear ordinary differential equations with constant coefficients

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy = 0 \quad (p, q = \text{const.}) \tag{11}$$

(to be specific, the second-order equation is given) is sought in the form

$$y = \exp(-\lambda x) \tag{12}$$

thus leading to the following algebraic (quadratic) equation for determining the exponent:

$$\lambda^2-p\lambda+q = 0. \tag{13}$$

The roots of equation (13) specify two linearly independent solutions of the initial differential equation (11).

Suppose that the condition $q < 0$ is fulfilled and equation (11) is solved under the following boundary conditions:

$$y(0) = 1, \quad y(\infty) = 0. \tag{14}$$

The solution of the problem (11) and (14) is given by expression (12) where λ is the positive root of the quadratic equation (13). In this case a local flux at the point $x = 0$ has the form

$$j = -\left(\frac{dy}{dx}\right)_{x=0} = \lambda. \tag{15}$$

It follows from this equality that the algebraic equation (13) is virtually an equation for determining a local flux.

We shall reformulate equivalently the two-point boundary problem (11) and (14) as an eigenvalue problem. For this, the independent variable in equation (11) will be stretched according to the formula

$$\xi = \lambda x. \tag{16}$$

This will yield

$$\lambda^2\frac{d^2y}{d\xi^2}+\lambda p\frac{dy}{d\xi}+qy = 0. \tag{17}$$

The solution of this equation, which depends on the parameter λ, will be sought with boundary conditions (14) and with an additional normalization-type requirement

$$\xi = 0, \quad dy/d\xi = -1. \tag{18}$$

The problem (17), (14) and (18) is equivalent to (11) and (14) and is virtually an eigenvalue problem. Namely, here, besides the solution $y = y(\xi)$, one should also find the eigenvalue of λ at which this solution is possible. In such an approach, the local heat flux will exactly coincide, by virtue of condition (18), with this eigenvalue of λ (equation (15)).

To illustrate the main idea of the method further it will be provisionally considered that the exact solution of problem (17), (18) and (14) is unknown. Moreover, it will be assumed that the main sought characteristic of the problem is the local heat flux, i.e. the eigenvalue of λ.

The eigenvalue of λ can be approximately determined in the following simple way not resorting to the solution of the differential equation (17). Namely, making use of the ideas behind the Galerkin method, multiply equation (17) by weighting function $w = w(\xi)$ and then integrate the expression obtained over ξ from zero to infinity assuming that all the integrals converge. This yields

$$\lambda^2A_2+\lambda pA_1+qA_0 = 0. \tag{19}$$

Here, the coefficients A_m are given by the expression

$$A_0 = \int_0^\infty wy\,d\xi, \quad A_m = \int_0^\infty w\left(\frac{d^my}{d\xi^m}\right)d\xi \tag{20}$$

where $m = 1, 2$.

An approximate algebraic equation for determining the eigenvalue of λ will be found from the integral identity (19) and (20) assuming that $y = \psi(\xi)$, where ψ is some prescribed function which satisfies boundary conditions (14) and (18). The values of the coefficients A_0, A_1, A_2 (equation (20)) are calculated and expression (19) turns out to be a quadratic equation for obtaining the unknown parameter λ.

Now, the coefficients of this approximate equation should be refined by asymptotic correction [1, 2] making use of the more simple, than initial, auxiliary problems (17), (18) and (14) in two limiting cases (for example, when $p \to 0$ and $p \to \infty$).

Accurate to the obvious redesignations $\lambda \to S$, $A_2/A_1 \to a_2$, $qA_0/A_1 \to a_0$, the approximate equation (19) coincides with the quadratic equation (7). Therefore, after the asymptotic correction the following equation results:

$$\lambda^2 - \lambda_\infty \lambda - \lambda_0^2 = 0 \tag{21}$$

which is obtained by substituting S with λ in equation (10). On the other hand, this very form can be obtained for the exact equation (13) if account is taken of the fact that when $p \to \infty$ and $p \to 0$, its asymptotics are given by the formulae $\lambda_\infty = p$ and $\lambda_0^2 = -q$. Therefore, in the considered case of the linear equation with constant coefficients (11) the above-mentioned procedure for approximate determination of the eigenvalue of λ leads to an exact result, equation (13).

2. PROBLEMS OF MASS AND HEAT TRANSFER DESCRIBED BY ORDINARY DIFFERENTIAL EQUATIONS

2.1. *Description of the algebraic method*

Two important points should first be outlined which will be required further. It follows from the comparison of the differential (17) and algebraic (19) equations that the latter may be obtained if all the differential terms and unknown functions are formally substituted by some constants. Account should also be taken of the fact that the final finding of the method of asymptotic correction is independent of the specific values of the coefficients A_m and therefore, it is even possible without loss in accuracy not to calculate them from equations (20) at the given functions $y = \psi(\xi)$ and $w = w(\xi)$ but, for simplicity, to take them directly to be equal to $A_m = \pm 1$.

The above-described simple means of deriving an approximate algebraic equation for a local flux (or, which is the same, for the eigenvalue of λ) is easily extended to much more complex non-linear ordinary differential equations with variable coefficients the exact solution of which is already unknown. Namely, an algebraic equation for a local flux can be directly obtained from the initial differential equation via the simple formal procedure: an independent variable is introduced according to equation (16), after which all of the dimensionless dominant parameters of the problem (for equation (11) these are p and q) and the quantity $\lambda = j$ is fixed at its locations, whereas the remaining terms (derivatives, functions and 'stretched' coordinates) are substituted by the constants which, without loss of generality, can be taken equal to unity.

This simple basic scheme substantially saves time and makes it possible to mentally deduce the sought approximate algebraic equation for a local flux. It results from the integration of the differential equation (obtained after the extension of equation (16) and multiplication by the weighting function $w(\xi)$) over the coordinate ξ from zero to infinity and subsequent assignment of some specific profile $y = \psi(\xi)$ with the use, at the final stage, of the method of asymptotic correction for the thus found algebraic equation. The errors, which are introduced in such an approximate approach, are attributable to the fact that the exact solution of the initial boundary-value problem, obtained after the substitution of equation (16), is generally dependent not only on the stretched coordinate ξ but also on the dimensionless parameters of the problem: $y = y(\xi; p, q)$.

As usual, in specific problems of convective mass and heat transfer and chemical hydrodynamics, the designation $y = c$ will be used for dimensionless concentration and equation (16) will be stretched by the formula

$$x = \xi/j \tag{22}$$

in which relation (15) has already been taken into account. When passing over from the stretched differential equation to the algebraic one, all the coordinates, functions and derivatives are substituted for $(+1)$ except for the first derivative $dy/d\xi$, which is replaced for convenience by the negative value (-1) in conformity with the physical implication, since under boundary conditions (14) the inequality $dy/d\xi < 0$ takes place.

Specific examples will now be considered which illustrate the possibilities of the method suggested.

2.2. *Turbulent fluid flow mass transfer compounded by a volumetric chemical reaction*

First, the following boundary-value problem for a second-order non-linear ordinary differential equation with variable coefficients will be considered:

$$\frac{d}{dx}\left[(1+\sigma x^n)\frac{dc}{dx}\right] - kf(c) = 0 \tag{23}$$

$$x = 0,\ c = 1; \quad x \to \infty; \quad c \to 0. \tag{24}$$

At $n = 3$, equation (23) describes the turbulent fluid flow mass transfer, compounded by a volumetric chemical reaction, near the tube walls far from the inlet section [3, 4]. In this particular case the dimensionless parameters and variables appearing in the formulation of problem (23), (24) are introduced by the formulae

$$x = \frac{X}{a}, \quad c = \frac{C}{C_s}, \quad k = \frac{a^2 KF(C_s)}{DC_s}, \quad f(c) = \frac{F(C)}{F(C_s)},$$

$$\sigma x^3 = \frac{D_T}{D}, \quad D_T = \gamma_T \left[\left(\frac{\tau_0}{\rho} \right)^{1/2} \frac{X}{\nu} \right]^3$$

where a is the tube radius; X the distance reckoned from the tube wall; C the time-averaged reagent concentration in the flow; C_s the concentration on the tube surface; K a constant of the volumetric chemical reaction rate; $W = KF(C)$ the volumetric reaction rate; D the diffusion coefficient; D_T the turbulent diffusion coefficient; τ_0 the shear stress on the tube wall; ρ and ν the density and kinematic viscosity of fluid, respectively; γ_T some empirical constant.

Henceforth, the value of the parameter n will not be specified. It will only be assumed that the condition $1 < n < \infty$ is fulfilled. An exact solution of problem (23) and (24) is unknown, therefore, it will be analysed by the algebraic method.

According to the method given above, the coordinate x in equation (23) will be stretched by equation (22). This will yield

$$j^2 \left\{ \frac{d^2 c}{d\xi^2} \right\} + \sigma j^{2-n} \left\{ \xi^n \frac{d^2 c}{d\xi^2} + n\xi^{n-1} \frac{dc}{d\xi} \right\} - k\{f(c)\} = 0. \quad (25)$$

Substitution of the expressions within braces, taking account of their sign, by the values ± 1 gives the algebraic equation for a diffusion flow

$$j^2 - \sigma j^{2-n} - k = 0. \quad (26)$$

Further assuming successively in equation (26) that $\sigma = 0$ and $k = 0$, the asymptotics of the quantity $j = j(\sigma, k)$ are found

$$j(0,k) = \sqrt{k}, \quad j(\sigma,0) = \sigma^{1/n}. \quad (27)$$

Eliminating from equations (27) the parameters σ and k in terms of $j(0,k)$ and $j(\sigma,0)$, equation (26) for a diffusion flow can be rewritten in the following form:

$$j^2(\sigma,k) - j^n(\sigma,0) j^{2-n}(0,k) - j^2(0,k) = 0. \quad (28)$$

To obtain the sought algebraic equation for j, it is necessary, from the initial differential equation (23) taking into account boundary conditions (24), to find exact values of the asymptotics $j(\sigma,0)$ and $j(0,k)$ and substitute them into relation (28).

At $\sigma = 0$, the solution of problem (23) and (24) can be written in an implicit form

$$x = \int_c^1 \left[2k \int_0^{c'} f(c'')\, dc'' \right]^{-1/2} dc'. \quad (29)$$

The diffusion flow $j = -(dc/dx)_{x=0}$, corresponding to relation (29), is determined by the formula

$$j(0,k) = (2k\langle f \rangle)^{1/2}, \quad \langle f \rangle = \int_0^1 f(c)\, dc. \quad (30)$$

In the other limiting case, at $k = 0$, the solution of problem (23) and (24) is given by the expression

$$c = \int_x^\infty \frac{dx}{1+\sigma x^n} \Big/ \int_0^\infty \frac{dx}{1+\sigma x^n} \quad (31)$$

thus leading to the following value for the diffusion flow:

$$j(\sigma,0) = \sigma^{1/n} \left(\frac{n}{\pi} \sin \frac{\pi}{n} \right). \quad (32)$$

The substitution of formulae (30) and (32) into equation (28) enables the following sought approximate algebraic equation for the diffusion flow to be derived:

$$j^2 - \sigma \left(\frac{n}{\pi} \sin \frac{\pi}{n} \right)^n j^{2-n} - 2k\langle f \rangle = 0. \quad (33)$$

In a specific case of the first order volumetric reaction, corresponding to linear equation (23) at $f = c$, the cubic equation (33) at $n = 3$ and $\langle f \rangle = 1/2$ was obtained in ref. [4] by another technique. A comparison was also made in this work between the root of this approximate equation and the data of direct numerical integration of the full initial two-point boundary-value problem (23) and (24) at $n = 3$, $f = c$ which showed a good agreement of these results for a diffusion flow (within 5%).

For the Nth-order volumetric reaction, which corresponds to $f(c) = c^N$, it should be assumed in equation (33) that $\langle f \rangle = (N+1)^{-1}$.

2.3. *Mass transfer to the surface of a disk rotating in a fluid in the course of a volumetric reaction*

Consider now the non-linear problem

$$\frac{d^2 c}{dx^2} + Px^m \frac{dc}{dx} = kf(c) \quad (34)$$

$$x = 0, c = 1; \quad x \to \infty, c \to 0 \quad (35)$$

which at $m = 2$ describes the concentration field in the vicinity of the disk rotating in a fluid. In this case dimensionless quantities are introduced by the formulae

$$x = \frac{X}{a}, \quad c = \frac{C}{C_s}, \quad P = 0.51 \frac{\nu}{D},$$

$$a = \left(\frac{\nu}{\omega} \right)^{1/2}, \quad kf(c) = \frac{a^2 KF(C)}{DC_s}$$

where X is the distance to the disk surface, ω the angular velocity of disk rotation.

Problem (34) and (35) also describes the concentration field in the vicinity of the forward stagnation point of a spherical droplet (at $m = 1$) and of a solid particle (at $m = 2$) immersed in a translational Stokes flow at large Peclet numbers (in these cases, the parameter P coincides with the Peclet number accurate to the constant factor).

To approximately analyse problem (34) and (35), use will be made of the algebraic method. After transformation of equation (22), equation (34) takes on the form

$$j^2\left\{\frac{d^2c}{d\xi^2}\right\}+Pj^{1-m}\left\{\xi^m\frac{dc}{d\xi}\right\}=k\{f(c)\}. \quad (36)$$

All the quantities within braces will be substituted for ± 1, and this will give the algebraic equation for the diffusion flow

$$j^2-Pj^{1-m}=k. \quad (37)$$

Now, asymptotic correction of equation (37) will be made by the limiting values of the parameter k. At $k=0$ equation (37) gives

$$j_0=P^{1/(m+1)} \quad (k=0). \quad (38)$$

For large k, when $m\geqslant 0$, equation (37) yields

$$j_\infty=\sqrt{k} \quad (k\to\infty). \quad (39)$$

Further, eliminating the quantities k and P from expressions (37)–(39) gives the following approximate equation:

$$j^2-j_0^{m+1}j^{m-1}-j_\infty^2=0. \quad (40)$$

It is into this formula that the corresponding asymptotics of the solution of the exact problem (34) and (35) should now be substituted.

At $k=0$, the solution of problem (34) and (35) has the form

$$c=\int_x^\infty \exp\left(-\frac{P}{m+1}x^{m+1}\right)dx\Bigg/\int_0^\infty \exp\left(-\frac{P}{m+1}x^{m+1}\right)dx.$$

Hence, the local flow will be given by

$$j_0=(m+1)^{1/(m+1)}\left[\Gamma\left(\frac{1}{m+1}\right)\right]^{-1}P^{1/(m+1)} \quad (41)$$

where $\Gamma(\alpha)$ is the gamma-function.

When $k\to\infty$, the second term on the left-hand side of equation (34) can be neglected. The solution of the corresponding 'truncated' problem is given by formula (29) and the diffusion flow is determined by the equation

$$j_\infty=(2k\langle f\rangle)^{1/2}, \quad \langle f\rangle=\int_0^1 f(c)\,dc. \quad (42)$$

Substituting the exact asymptotics (41) and (42) into equation (40) gives the sought approximate algebraic equation for the diffusion flow

$$j^2-(m+1)^m\left[\Gamma\left(\frac{1}{m+1}\right)\right]^{-m-1}Pj^{1-m}-2k\langle f\rangle=0. \quad (43)$$

It follows from the results of Section 1 that in the case of the first-order volumetric chemical reaction ($f=c$, $\langle f\rangle=1/2$) at $m=0$ equation (43) is accurate.

Remark 1. When deriving approximate algebraic equations, it is possible not to take into account the sign of the first derivative. In other words, all the terms, without exception, omitted in the initial differential equation can be substituted by unity (i.e. there is no need to keep additional information in memory). The subsequent procedure of asymptotic correction by all means will automatically lead to a correct result. For example, having omitted the terms within braces of differential equation (36), it is possible to obtain the following equation instead of equation (37):

$$j^2+Pj^{1-m}=k \quad (37a)$$

(here the sign of the first derivative has been disregarded already). Letting the parameter k in this equation approach zero yields

$$P=-j_0^{m+1} \quad (k=0). \quad (38a)$$

In the other limiting case, when $k\to\infty$, asymptotic (39) is obtained, with the inequality $m\geqslant 0$ being taken into account.

Eliminating the parameters P and k from expressions (38a) and (39) in terms of j_0 and j_∞ and further substituting them into formula (37a) give the same algebraic equation (40).

It will be shown now in what way the above method can be applied to approximately analyse linear parabolic equations the coefficients of which are independent of time, and particular problems will be considered.

2.4. *Non-stationary mass exchange of a wall with a quiescent medium in the course of a first-order volumetric reaction*

Mass exchange of a wall with a quiescent medium compounded by a first-order reaction is described by the following dimensionless equation, initial and boundary conditions:

$$\frac{\partial c}{\partial \tau}=\frac{\partial^2 c}{\partial x^2}-kc \quad (44)$$

$$\tau=0,\ c=0; \quad x=0,\ c=1; \quad x\to\infty,\ c\to 0. \quad (45)$$

Making use of the Laplace–Carson transform

$$\bar{c}=p\int_0^\infty e^{-p\tau}c\,d\tau \quad (46)$$

reduce the problem for the partial differential equation (44) and (45) to the problem for the ordinary differential equation with the complex parameter p

$$\frac{d^2\bar{c}}{dx^2}=(p+k)\bar{c} \quad (47)$$

$$x=0,\ \bar{c}=1; \quad x\to\infty,\ \bar{c}\to 0. \quad (48)$$

Similar to equation (22), substitute $x=\xi/\bar{j}$ in equation (47), where $\bar{j}$ is the image of the diffusion flow. As a result, obtain

$$\bar{j}^2\left\{\frac{d^2\bar{c}}{d\xi^2}\right\} = (p+k)\{\bar{c}\}. \quad (49)$$

In accordance with the algebraic method omit the quantities within braces. This gives a quadratic equation for $\bar{j}$ the solution of which has the form

$$\bar{j} = (p+k)^{1/2}. \quad (50)$$

It is easy to verify that expression (50) does not need asymptotic correction, because it gives accurate asymptotics in the limiting cases when $p \to 0$ and $p \to \infty$. Therefore, applying the inverse Laplace–Carson transform to equation (50) [5], the formula for the diffusion flow can be found

$$j = \frac{1}{\sqrt{(\pi\tau)}}\exp(-k\tau)+\sqrt{k}\,\mathrm{erf}\sqrt{(k\tau)}. \quad (51)$$

It is not difficult to show that expression (51) coincides with the results of the exact solution of the initial problem (44) and (45).

2.5. *Mass and heat transfer in turbulent flow in the tube inlet section*

Consider a turbulent flow with constant physical properties in a round tube of radius a. Assume that the admixture concentration in the initial section at $Z = 0$ is equal to C_∞ (Z-axis is directed along the tube axis), whereas on the tube wall at $X = 0$ it is equal to C_s. At large Schmidt numbers the main resistance to mass transfer is concentrated in a thin diffusion boundary layer adjacent to the tube surface. In this region, the distribution of fluid velocities can be taken linearly dependent on the distance to the tube walls and the normal component of the turbulent diffusion can be described by the main term of expansion when $X \to 0$.

Under the assumptions made the distribution of the time-averaged concentration in the tube inlet section is described by the following equation and boundary conditions [6]:

$$x\frac{\partial c}{\partial z} = \frac{\partial}{\partial x}(1+\sigma x^n)\frac{\partial c}{\partial x} \quad (52)$$

$$z = 0, c = 0;\quad x = 0, c = 1;\quad x \to \infty, c \to 0 \quad (53)$$

where

$$c = \frac{C_\infty - C}{C_\infty - C_s},\quad z = \frac{aD(\langle u\rangle)^2 Z}{8\nu^3}$$

$\langle u\rangle$ is the section-averaged fluid velocity; the rest of the dimensionless values are introduced analogously to equation (23).

Application of the Laplace–Carson transform (46) (in which τ is substituted by z) to problem (52) and (53), gives the ordinary differential equation

$$\frac{d}{dx}(1+\sigma x^n)\frac{d\bar{c}}{dx} = px\bar{c} \quad (54)$$

$$x = 0, \bar{c} = 1;\quad x \to \infty, \bar{c} \to 0. \quad (55)$$

In accordance with the algebraic method, stretch equation (54) by the formula $x = \xi/\bar{j}$, where $\bar{j}$ is the diffusion flow image. As a result, obtain

$$\bar{j}^3\left\{\frac{d^2\bar{c}}{d\xi^2}\right\}+\sigma\bar{j}^{3-n}\left\{\xi^n\frac{d^2\bar{c}}{d\xi^2}+n\xi^{n-1}\frac{d\bar{c}}{d\xi}\right\} = p\{\xi\bar{c}\}. \quad (56)$$

Omitting the terms within braces gives the algebraic equation with the complex parameter p

$$\bar{j}^3 - \sigma\bar{j}^{3-n} - p = 0. \quad (57)$$

At $n = 3$ the solution of equation (57) is given by

$$\bar{j} = (\sigma+p)^{1/3}. \quad (58)$$

Application of the inverse Laplace–Carson transform to equation (58) [5] gives the diffusion flow

$$j = \frac{\sigma^{1/3}}{\Gamma(2/3)}[\gamma(\tfrac{2}{3}, \sigma z)+(\sigma z)^{-1/3}\exp(-\sigma z)] \quad (59)$$

where

$$\gamma(\alpha, x) = \int_0^x \zeta^{\alpha-1}e^{-\zeta}\,d\zeta$$

is the incomplete gamma-function, $\Gamma(\alpha) = \gamma(\alpha, \infty)$ is the gamma-function.

This formula should be refined by the method of asymptotic correction. Asymptotics (59) at small and large z have the form

$$j_0 = \frac{1}{\Gamma(2/3)z^{1/3}}(z \to 0);\quad j_\infty = \sigma^{1/3}\ (z \to \infty). \quad (60)$$

Elimination of z and σ from equations (59) and (60) yields

$$\frac{j}{j_\infty} = \frac{1}{\Gamma(2/3)}\gamma(\tfrac{2}{3}, A)+\frac{j_0}{j_\infty}e^{-A} \quad (61)$$

where

$$A = \frac{1}{[\Gamma(2/3)]^3}\frac{j_\infty^3}{j_0^3}.$$

Now find the exact asymptotics of the solution to the initial problem (52) and (53) at $n = 3$.

When $z \to \infty$, the left-hand side of equation (52) can be neglected. The solution of the corresponding 'truncated' problem is given by formula (31). At $n = 3$ it leads to the following expression for the limiting diffusion flow:

$$j_\infty = \frac{3^{3/2}}{2\pi}\sigma^{1/3}\quad (z \to \infty). \quad (62)$$

It can be shown that at small z the second term on the right-hand side of equation (52) is small as compared with the first one. Therefore, to find the sought asymptotic it is necessary to consider the equation

$$x\frac{\partial c}{\partial z}=\frac{\partial^2 c}{\partial x^2} \quad (63)$$

with boundary conditions (53). The solution of this problem is given by

$$c=\frac{1}{\Gamma(1/3)}\Gamma\left(\frac{1}{3},\frac{x^3}{9z}\right). \quad (64)$$

Differentiating expression (64) with respect to x and assuming $x = 0$, it is possible to calculate the diffusion flow

$$j_0=\frac{3^{1/3}}{\Gamma(1/3)}\frac{1}{z^{1/3}} \quad (z\to 0). \quad (65)$$

In accordance with the algebraic method, the exact asymptotics (62) and (65) should be substituted into formula (61). The resulting approximate relation $j = j(z)$, which is not written out here, will give an accurate result in the limiting cases when $z \to 0$ and $z \to \infty$.

3. PROBLEM OF A STATIONARY DIFFUSION BOUNDARY LAYER (PARTIAL DIFFERENTIAL EQUATIONS)

3.1. *Description of the algebraic method*

In those cases when the necessity arises to study stationary boundary layer equations, described by partial differential equations, the proposed method for deriving approximate algebraic equations to determine the integral mean characteristics of the problem virtually does not change and remains the same as for ordinary differential equations. In this case the stretching of the boundary layer coordinate x, directed normal to the body surface, should be made by equation (22), where, by virtue of the analogy principle [2], the local flux j can be immediately replaced by the mean Sherwood number, i.e. it can be assumed that

$$x=\frac{\xi}{Sh}. \quad (66)$$

After the above mentioned transformation, the dimensionless parameters of the problem and the sought value Sh, will be retained whereas the remaining terms will be formally assumed, as before, to be equal to ± 1 depending on their sign. The fulfillment of these operations gives an approximate algebraic equation for the sought quantity Sh, which should be improved by the method of asymptotic correction. It should be especially emphasized that here, in derivation of the algebraic equation, the additional dependence of the equation coefficients and differential operators on the longitudinal coordinates directed along the body surface is disregarded (the corresponding terms are replaced by unity); thus the lost information will be partially compensated for further as a result of the use of the asymptotic correction technique thus leading to the refinement of the coefficients of the unknown approximate equation for mass and heat transfer coefficients.

The algebraic method described above with reference to the partial differential equations is equivalent in a number of cases to the sequential accomplishment of the following three stages: first, a degenerate auxiliary problem on the distribution of concentration near a singular point of incidence is formulated which is already described by an ordinary differential equation (i.e. actually, the second-order model problem is considered at the first stage, see ref. [2]); then, this auxiliary problem is solved by the Galerkin method; at the last stage the auxiliary relation obtained at the intermediate step for the local flow is appropriately rewritten in terms of asymptotics, thus finally leading, with the aid of the similarity principle [1, 2], to the desired approximate equation for the mean Sherwood number.

3.2. *Diffusion to a droplet at large Peclet numbers within the entire range of phase viscosities*

Consider a stationary convective diffusion to the surface of a spherical droplet of radius a in a translational Stokes flow (Hadamard–Rybchinsky flow) at large Peclet numbers. It is assumed that the concentration on the droplet surface and far from it are constant and equal to C_s and C_∞, respectively. In dimensionless variables and in the spherical coordinate system r, θ, fixed in the droplet, the corresponding boundary-value problem (written in the diffuse boundary layer approximation) is [2]

$$-2\cos\theta(\lambda_1 x+\lambda_2 x^2)\frac{\partial c}{\partial x}+\sin\theta(\lambda_1+2\lambda_2 x)\frac{\partial c}{\partial\theta}=\frac{1}{Pe}\frac{\partial^2 c}{\partial x^2} \quad (67)$$

$$x=0,\ c=1;\quad x\to\infty,\ c\to 0(\theta=0,\partial c/\partial\theta=0) \quad (68)$$

$$c=\frac{C_\infty-C}{C_\infty-C_s},\quad x=r-1,\quad Pe=\frac{aU_\infty}{D},$$

$$\lambda_1=\frac{1}{2(\beta+1)},\quad \lambda_2=\tfrac{3}{4}. \quad (69)$$

Here C is the concentration in the flow, D the coefficient of diffusion, U_∞ the flow velocity far from the droplet, β the ratio between the dynamic viscosities of the droplet and surrounding fluid ($\beta = \infty$ corresponds to a solid particle and $\beta = 0$ to a gas bubble).

Note that in equation (69) instead of the exact value of the second coefficient $\lambda_2 = (3\beta+2)/(4\beta+4)$ its asymptotic for $\beta \to \infty$ is taken. The validity of this simplification of the problem at large Peclet numbers is proved in ref. [2].

To analyse problem (67)–(69), use is made of the algebraic method. For this purpose equation (66) is substituted into equation (67). This will yield

$$\frac{Pe}{\beta+1}\left\{-\xi\cos\theta\frac{\partial c}{\partial\xi}+\frac{1}{2}\sin\theta\frac{\partial c}{\partial\theta}\right\}+\frac{Pe}{Sh}\times\left\{-\frac{3}{2}\xi^2\cos\theta\frac{\partial c}{\partial\xi}+\frac{3}{2}\xi\sin\theta\frac{\partial c}{\partial\theta}\right\}=Sh^2\left\{\frac{\partial^2 c}{\partial\xi^2}\right\}. \quad (70)$$

Replacing all the terms within braces in equation (70) by ± 1 gives a cubic equation

$$Sh^3-\frac{Pe}{\beta+1}Sh-Pe=0. \quad (71)$$

Taking into account the fact that the condition $Pe \gg 1$ is fulfilled, consider two limiting cases: $\beta \sim 1$ and $\beta=\infty$. In the first case which corresponds to a droplet with moderate viscosity, the last term in equation (71) can be neglected when $Pe \to \infty$. This leads to the asymptotic

$$Sh_\beta=\left(\frac{Pe}{\beta+1}\right)^{1/2} \quad (\beta \sim 1). \quad (72)$$

At $\beta=\infty$, corresponding to a solid particle, obtain from equation (71)

$$Sh_\infty = Pe^{1/3}. \quad (73)$$

Elimination of the parameters β and Pe from expressions (71)–(73) yields the sought cubic equation for the mean Sherwood number

$$Sh^3-Sh_\beta^2\,Sh-Sh_\infty^3=0 \quad (74)$$

which was derived earlier in refs. [1, 2] by a different technique.

It is necessary to substitute into equation (74) the exact asymptotic solutions of problem (67) and (68) which were obtained at

$$\lambda_1=\frac{1}{2(\beta+1)},\quad \lambda_2=0 \quad\text{and}\quad \lambda_1=0,\quad \lambda_2=\tfrac{3}{4}$$

in ref. [7]

$$Sh_\beta=\left[\frac{Pe}{3\pi(\beta+1)}\right]^{1/2},\quad Sh_\infty=0.624Pe^{1/3}. \quad (75)$$

In ref. [2] the approximate formula (75) is compared with the results of numerical calculations; the comparison showed its high accuracy at large Peclet numbers within the entire range of change of the parameter β.

3.3. *Mass exchange between a droplet and a flow compounded by a volumetric reaction*

The following non-linear equation is now studied:

$$\frac{Pe}{\beta+1}\left(-x\cos\theta\frac{\partial c}{\partial x}+\frac{1}{2}\sin\theta\frac{\partial c}{\partial\theta}\right)=\frac{\partial^2 c}{\partial x^2}-kf(c) \quad (76)$$

which, together with boundary conditions (35), describes the distribution of concentration in the diffuse boundary layer of the droplet in the course of a volumetric reaction which takes place in the phase and the rate of which depends arbitrarily on concentration.

Making use of equation (66), transform equation (76) into

$$\frac{Pe}{\beta+1}\left\{-\xi\cos\theta\frac{\partial c}{\partial\xi}+\frac{1}{2}\sin\theta\frac{\partial c}{\partial\theta}\right\}=Sh^2\left\{\frac{\partial^2 c}{\partial\xi^2}\right\}-k\{f(c)\}.$$

Omitting further the terms within braces, obtain a quadratic equation

$$Sh^2=\frac{Pe}{\beta+1}+k. \quad (77)$$

Allowing for the fact that the asymptotics of equation (77) have the form

$$Sh_0=\left(\frac{Pe}{\beta+1}\right)^{1/2}(k\to 0);\quad Sh_\infty=k^{1/2}(k\to\infty)$$

the solution of equation (77) will be represented in the following form:

$$Sh=(Sh_0^2+Sh_\infty^2)^{1/2}. \quad (78)$$

The exact asymptotic of the solution of problem (76) and (35) for $k\to 0$ and $k\to\infty$ should be substituted into the latter expression. At $k=0$ equation (76) passes over to equation (67) at $\lambda_2=0$; its solution leads to the Sherwood number $Sh=Sh_\beta$ the right-hand side of which is written out in equation (75). When $k\to\infty$, the left-hand side of equation (76) is inessential, and the corresponding solution can be represented in an implicit form (29). In this limiting case the Sherwood number will be determined by the right-hand side of equation (30).

Substituting the above-mentioned asymptotics into equation (78), find the sought relation for the Sherwood number

$$Sh=\left[\frac{Pe}{3\pi(\beta+1)}+2k\int_0^1 f(c)\,dc\right]^{1/2}. \quad (79)$$

It follows from the comparison [8] of equation (79) with the results of the exact solution of problem (76) and (35) for the first-order reaction that the maximum error of equation (79) at $f(c)=c$ is around 7% within the entire range of change of the parameter $k(0\leqslant k<\infty)$.

3.4. *Mass exchange between a solid particle and a flow compounded by a volumetric reaction*

Consider the non-linear equation

$$Pe\left(-\frac{3}{2}x^2\cos\theta\frac{\partial c}{\partial x}+\frac{3}{2}x\sin\theta\frac{\partial c}{\partial\theta}\right)=\frac{\partial^2 c}{\partial x^2}-kf(c) \quad (80)$$

which together with boundary conditions (35) describes the concentration field in the diffuse boundary layer of a solid spherical particle in a translational Stokes flow. It is assumed that a volumetric chemical reaction takes place in the liquid.

The substitution of equation (66) leads equation (80) to the form

$$\frac{Pe}{Sh}\left\{-\frac{3}{2}\xi^2\cos\theta\frac{\partial c}{\partial\xi}+\frac{3}{2}\xi\sin\theta\frac{\partial c}{\partial\theta}\right\} = Sh^2\left\{\frac{\partial^2 c}{\partial\xi^2}\right\}-k\{f(c)\}.$$

Omitting the terms within braces gives the cubic equation

$$Sh^3-k\,Sh-Pe=0. \quad (81)$$

It can be easily verified that equation (81) can be rewritten as

$$Sh^3-Sh_\infty^2\,Sh-Sh_0^3=0 \quad (82)$$

where Sh_0 and Sh_∞ are the asymptotic solutions of equation (81) for $k\to 0$ and $k\to\infty$.

It is necessary now to substitute the corresponding exact asymptotic solutions of problem (80) and (35) into equation (82). At $k=0$, equation (80) coincides with equation (67) at $\lambda_1=0$; its solution gives the Sherwood number $Sh_0=0.624Pe^{1/3}$. When $k\to\infty$, the Sherwood number will be prescribed just as in the case of a droplet, by the expression $Sh_\infty=(2k\langle f\rangle)^{1/2}$.

Substituting these exact asymptotics into equation (82), obtain the sought cubic equation for the mean Sherwood number

$$Sh^3-2k\langle f\rangle Sh-\alpha\,Pe=0 \quad (83)$$

where $\alpha=0.243\approx(0.624)^3$.

4. TWO MODIFICATIONS OF THE ALGEBRAIC METHOD OF STUDYING NON-STATIONARY PROBLEMS (PARTIAL DIFFERENTIAL EQUATIONS)

The algebraic method can also be successfully used to solve non-stationary problems when the characteristics of the process differ greatly with time. In this case the main formal scheme of method application admits two different modifications. First, each of these will be briefly described and then the results of the solution of a number of specific non-stationary problems by both methods will be analysed and compared.

It should be noted that, in both the first and second modifications, preliminarily the coordinate x is stretched according to rule (22) with subsequent fixation of all the dimensionless parameters of the problem and of the diffusion flow at their places and substitution of the remaining terms, which incorporate the derivatives with respect to coordinates and the functions of dependent and independent variables, by the values ± 1 (according to their sign), just as it was done earlier in stationary problems. The approaches differ only in the way of the substitution of the non-stationary term which involves a partial derivative with respect to time $\partial c/\partial\tau$, where τ is the dimensionless time. In the limiting case of infinitely large times (when $\tau\to\infty$) both modifications lead to identical results corresponding to the solution of the same stationary problem.

4.1. *The first modification of the algebraic method*

It involves the formal substitution of the partial derivative with respect to time in a differential equation by the algebraic relation according to the following rule:

$$\frac{\partial c}{\partial\tau}\Rightarrow\frac{1}{\tau}. \quad (84)$$

When refining the coefficients of the thus obtained algebraic equation by the method of asymptotic correction, the dimensionless time is already considered to be a new additional parameter (i.e. when improving the algebraic expression, use is made of the exact asymptotics of the initial problem in the limiting cases $\tau\to 0$ and $\tau\to\infty$).

In other words, the first modification of the algebraic method consists in the sequential fulfillment of three operations: first, the derivative $\partial c/\partial\tau$ in the differential equation is substituted according to rule (84); then the coordinate x is stretched by equation (22); after this all of the initial dimensionless parameters of the problem and the additional quantities j and τ are fixed at their places and the rest of the terms are replaced by ± 1. A simple physical explanation for this formal technique of obtaining approximate algebraic equations will be presented in Section 4.2 (see Remark 2).

4.2. *The second modification of the algebraic method*

The second modification of the algebraic method is somewhat more complex. It consists of a formal substitution of the partial time derivative by the following complex which involves the 'direct' derivative

$$\frac{\partial c}{\partial\tau}\Rightarrow-\frac{1}{j}\frac{dj}{d\tau}. \quad (85)$$

This, at first sight rather complicated rule can be easily explained when it is remembered that there is a complete identity between the algebraic method and the combination of the Galerkin method with the method of asymptotic correction in application to stationary problems described by ordinary differential equations. Now, requiring the fulfillment of the equivalency of the above methods also more complex non-stationary problems, which are formulated for partial differential equations it is possible to show that this requirement will lead to the above correspondence (85).

In fact, in the integral methods the solution of non-stationary problems is sought in the form $c=\psi(x/\delta)$, where $\delta = \delta(\tau)$ is the boundary layer thickness which already depends on time. For the initial and boundary conditions (45) under consideration the concentration profile ψ and the function δ should satisfy the conditions $\psi(0) = 1$, $\psi(\infty) = 0$, $\delta(0) = 0$. In accordance with the Galerkin method multiply the derivative $\partial c/\partial \tau$ by any weighting function of the type $w = w(x/\delta) > 0$. Integration of the resulting expression over x from zero to infinity leads to the following set of equalities:

$$\int_0^\infty w\left(\frac{x}{\delta}\right)\frac{\partial c}{\partial \tau}\,\mathrm{d}x = \int_0^\infty w\left(\frac{x}{\delta}\right)\frac{\partial}{\partial \tau}\left[\psi\left(\frac{x}{\delta}\right)\right]\mathrm{d}x$$

$$= \frac{\mathrm{d}\delta}{\mathrm{d}\tau}\int_0^\infty \zeta w(\zeta)\left(-\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)\mathrm{d}\zeta = \frac{\text{const.}}{j}\left(-\frac{1}{j}\frac{\mathrm{d}j}{\mathrm{d}\tau}\right) \quad (86)$$

where the relation $j = \text{const.}\ \delta^{-1}$ was taken into account. It is assumed that the remaining part of the equation is a sum of the terms of the form $f(c)x^n(\partial^m c/\partial x^m)$. Being multiplied by the weighting function w and integrated over x from 0 to ∞, these quantities are transformed by

$$\int_0^\infty w\left(\frac{x}{\delta}\right)f(c)x^n\left(\frac{\partial^m c}{\partial x^m}\right)\mathrm{d}x$$

$$= \delta^{n+1-m}\int_0^\infty w(\zeta)\zeta^n f(\psi(\zeta))\frac{\mathrm{d}^m\psi}{\mathrm{d}\zeta^m}\,\mathrm{d}\zeta$$

$$= \frac{\text{const.}}{j}(j^{m-n}). \quad (87)$$

It is precisely the comparison of expressions (86) and (87) with regard for the method based on the stretching of equation (22) for stationary problems (see Section 2), and also the use of the fact that the specific values of constants in the initial approximate formula are inessential for the method of asymptotic correction which leads to rule (85).

From the foregoing, it is clear that the above modification of the algebraic method for obtaining approximate relations includes three stages: first, a non-stationary term of the partial differential equation is substituted by a differential term according to rule (85); then the spatial coordinate is stretched by formula (28) and, finally, all of the dimensionless parameters and quantities j and $\mathrm{d}j/\mathrm{d}\tau$ are fixed at their places whereas the remaining factors are substituted by the constants equal to ± 1. After that, the thus obtained differential equation is solved. At the third stage, according to general rules, the solution is refined by the asymptotic correction method; now the time τ enjoys already the same right as do the dimensionless initial parameters of the problem.

It follows from the comparison of expressions (84) and (85) that the transition from the algebraic equation for a diffusion flow, obtained by the first modification of the method, to an ordinary differential equation, derived by the second modification of the method (and conversely), is accomplished by means of the simple correspondence

$$\frac{1}{\tau} \Leftrightarrow -\frac{1}{j}\frac{\mathrm{d}j}{\mathrm{d}\tau}. \quad (88)$$

This rule of transition is most useful when applied at the final stage after asymptotic correction of the algebraic equation coefficients thus making it possible to directly obtain a differential equation which gives a correct result in the limiting case $\tau \to \infty$.

The approximate solutions obtained by both modifications of the algebraic method coincide at small and large times. However, there is one substantial qualitative difference in them: the solution of the algebraic equation, which corresponds to the first modification of the method, decreases in the power-law fashion at large times, whereas the solution of the differential equation derived by the second modification of the method damps out much quicker exponentially.

The now available exact solutions of non-stationary mass and heat transfer problems have a decreasing exponential character with $\tau \to \infty$. This important fact suggests that the second modification of the algebraic method leads to more accurate results. Nevertheless, it will be shown further on specific examples, that the first modification of the method also gives good results which can be used directly in engineering practice.

Remark 2. Some considerations will now be given which were involved to formulate the first modification of the algebraic method. It is assumed that the Galerkin method gives the equation for the boundary layer thickness δ. Let the quantity δ be limited at large times

$$\lim_{\tau \to \infty} \delta \neq \infty. \quad (89)$$

The requirement that $\delta(0) = 0$ at small times gives the following approximate equality for the derivative $\mathrm{d}\delta/\mathrm{d}\tau$:

$$\frac{\mathrm{d}\delta}{\mathrm{d}\tau} \approx \frac{\delta}{\tau} \quad (\tau \to 0). \quad (90)$$

On the other hand, by virtue of equation (89) at large times the ratio δ/τ and the derivative $\mathrm{d}\delta/\mathrm{d}\tau$ have the same asymptotics

$$\frac{\mathrm{d}\delta}{\mathrm{d}\tau} \to 0, \quad \frac{\delta}{\tau} \to 0 \quad (\tau \to \infty). \quad (91)$$

Allowing for the fact that in both limiting cases, $\tau \to 0$ and $\tau \to \infty$, the quantities $\mathrm{d}\delta/\mathrm{d}\tau$ and δ/τ behave identically, it is possible in the equation for the boundary layer thickness to substitute the differential term by the algebraic one according to the rule

$$\frac{\mathrm{d}\delta}{\mathrm{d}\tau} \Rightarrow \frac{\delta}{\tau}. \quad (92)$$

The use of the relationship between the boundary

Table 1. Master table of the main rules of the algebraic method operation

Term in the initial differential equation (parabolic type)	Rule of substitution in the derivation of an approximate equation for a diffusion flow j
$\dfrac{\partial c}{\partial \tau}$	$\dfrac{1}{\tau}$ the first modification $-\dfrac{1}{j}\dfrac{\mathrm{d}j}{\mathrm{d}\tau}$ the second modification
$\dfrac{\partial^m c}{\partial x^m}$	$(-j)^m$
x^n	$1/j^n$
$f(c)$	1
$g(\tau)$	$g(\tau)$
k characteristic parameter of the problem (used for asymptotic correction)	k

Note. When derived, the approximate algebraic (differential) equation for the diffusion flow j should be refined by the method of asymptotic correction.

layer thickness and the diffusion flow $\delta = \text{const.}/j$ makes it possible to obtain from equation (92) that

$$-\frac{1}{j^2}\frac{\mathrm{d}j}{\mathrm{d}\tau} \Rightarrow \frac{1}{\tau j}. \tag{93}$$

Correspondence (93), which is analogous to correspondence (88), allows one to pass over from the second to the first modification of the algebraic method. Therefore, the first modification of the algebraic method is equivalent to the successive use of the Galerkin method, approximate substitution of the derivative rule (92), transition from the boundary layer thickness δ to a diffusion flow by the formula $j = \text{const.}/\delta$, and to the application of asymptotic correction at the final stage of the method.

4.3. *Master table of the main rules of the algebraic method*

Taking into consideration the foregoing, it is not difficult to compile a simple master table of the main rules of the algebraic method operation (see Table 1) which makes it possible to quickly derive the sought algebraic equations for the diffusion flow j. For this purpose, each term of the initial differential equation, which corresponds to a certain combination of expressions in the left-hand column of the table, should be substituted by the corresponding quantity located in the right-hand column. The resulting algebraic (for the first modification of the algebraic method) or differential (for the second modification) equation for a diffusion flow should be then refined by the method of asymptotic correction.

Differential equations obtained with the aid of the second modification of the algebraic method should be supplemented with the initial condition: $j(0) = \infty$.

Now, it will be shown in which way Table 1 of the main rules of the algebraic method operation should be used for solving specific non-stationary problems.

5. SOLUTION OF SPECIFIC MASS AND HEAT TRANSFER PROBLEMS WITH THE AID OF THE ALGEBRAIC METHOD

5.1. *Non-stationary mass exchange between a wall and a quiescent medium during a volumetric reaction the rate of which depends arbitrarily on concentration*

Consider now a more general, than (44) and (45), non-linear problem which is described by

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} - kf(c) \tag{94}$$

with the initial and boundary conditions (45). In equations (94) and (45) the dimensionless quantities are related to dimensional ones in the following way:

$$c = \frac{C}{C_s}, \quad \tau = \frac{Dt}{a^2}, \quad x = \frac{X}{a},$$

$$k = \frac{a^2 KF(C_s)}{DC_s}, \quad f(c) = \frac{F(C)}{F(C_s)}$$

where C is the concentration, C_s the concentration on the wall surface, t the time, D the coefficient of diffusion, X the distance to the wall, a the quantity which has the dimensional representation of length, K the constant of the volumetric chemical reaction rate, $W = KF(C)$ the volumetric reaction rate.

Substituting each term of the differential equation (94) according to the rules given in Table 1 for the first modification of the algebraic method yields the equation for determining the dependence of the diffusion flow on time

$$\frac{1}{\tau} = j^2 - k. \quad (95)$$

The coefficients of this expression will be refined with the use of the asymptotic correction method. In the limit $\tau \to \infty$, equation (95) gives

$$j_\infty = \sqrt{k} \quad (\tau \to \infty). \quad (96)$$

At small times equation (95) yields

$$j_0 = 1/\sqrt{\tau} \quad (\tau \to 0). \quad (97)$$

Further, eliminating the parameters k and τ from equation (95) with the aid of the limiting relations (96) and (97), a formula is obtained for a diffusion flow

$$j = (j_0^2 + j_\infty^2)^{1/2}. \quad (98)$$

Now, exact asymptotic solutions will be obtained for the initial problem (94) and (95) at small and large times. When $\tau \to 0$, the second term on the right-hand side of equation (94) can be neglected, and this, with the initial and boundary conditions (45), leads to the well-known distribution of concentration

$$c = \operatorname{erf} c\left(\frac{x}{2\sqrt{\tau}}\right). \quad (99)$$

This expression gives the following relation for a diffusion flow:

$$j_0 = (\pi\tau)^{-1/2} \quad (\tau \to 0). \quad (100)$$

In the other limiting case $\tau \to \infty$ the left-hand side of equation (94) is inessential and the corresponding asymptotic for j is given by equation (42).

Substituting relations (42) and (100) into equation (98), obtain the sought expression for a diffusion flow

$$j = \left(\frac{1}{\pi\tau} + 2k\langle f\rangle\right)^{1/2}, \quad \langle f\rangle = \int_0^1 f(c)\,\mathrm{d}c. \quad (101)$$

In specific cases of the first- and second-order reactions, which correspond to $f = c$ and c^2, formula (101) was derived by a different technique in ref. [4].

In this case the second modification of the algebraic method leads, according to Table 1, to the ordinary differential equation

$$-\frac{1}{j}\frac{\mathrm{d}j}{\mathrm{d}\tau} = j^2 - k \quad (102)$$

the solution of which has the form

$$j = \sqrt{k}[1 - \exp(-2k\tau)]^{-1/2}. \quad (103)$$

To refine the coefficients of this expression, use is made of the asymptotic correction method. When $\tau \to \infty$, equation (103) gives asymptotic (96). At small times equation (103) yields

$$j_0 = (2\tau)^{-1/2} \quad (\tau \to 0). \quad (104)$$

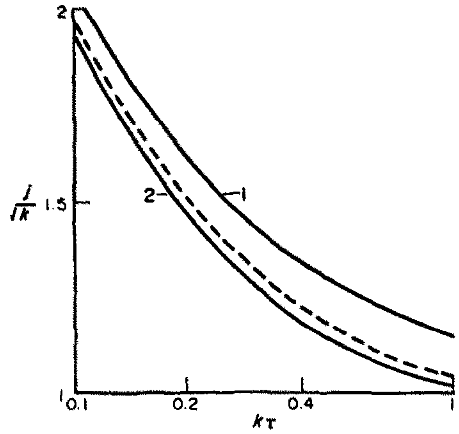

FIG. 2. Comparison of approximate (101), (106) (solid lines 1, 2), and exact (51) (dashed line) variation of a diffusion flow with time for the first-order reaction.

Eliminating the parameters k and τ from formula (103) by the limiting relations (96) and (104) gives the following expression:

$$\frac{j}{j_\infty} = \left[1 - \exp\left(-\frac{j_\infty^2}{j_0^2}\right)\right]^{-1/2}. \quad (105)$$

Substituting now the exact asymptotics (42) and (100) into equation (105), find the approximate dependence of the diffusion flow on time

$$j = (2k\langle f\rangle)^{1/2}[1 - \exp(-2\pi k\langle f\rangle\tau)]^{-1/2}. \quad (106)$$

Figure 2 presents a comparison of the approximate (101) and (106) (solid lines) and exact (51) (a dashed line) relations for the first-order reaction (which corresponds to $f = c$ and $\langle f\rangle = 1/2$). It is seen that expression (101) obtained with the use of the first modification of the algebraic method leads to overestimated results as compared with the exact solution. The maximum difference of the approximate formula (101) from the exact equation (51) amounts to around 10%. As follows from Fig. 2, the approximate expression (106) derived with the aid of the second modification of the algebraic method gives somewhat underestimated values as compared with the exact result. The error amounts in this case to about 4%.

5.2. *Mass and heat transfer in turbulent flow in the tube inlet section*

Let us return now to problem (52) and (53) in which the longitudinal coordinate serves as the time analogue. According to the first modification of the method Table 1 (at $\tau = z$) gives the following algebraic equation for a diffusion flow:

$$\frac{1}{zj} = j^2 - \sigma j^{2-n}. \quad (107)$$

Find the asymptotical values of j at small and large z.

Taking into account that $n > 0$, find from expression (107) for $z \to 0$

$$j_0 = z^{-1/3} \quad (z \to 0). \tag{108}$$

In the other limiting case

$$j_\infty = \sigma^{1/n} \quad (z \to \infty). \tag{109}$$

Eliminating the quantities z and σ in formula (107) with the aid of asymptotics (108) and (109) and making simple transformations give the sought algebraic equation for a diffusion flow

$$j^n - j_0^3 j^{n-3} - j_\infty^n = 0. \tag{110}$$

In accordance with the approach used, the quantities j_0 and j_∞ in equation (110) should be substituted by the exact asymptotic solution of the initial problem (52) and (53) for $z \to 0$ and $z \to \infty$ which are given by equations (65) and (32), respectively. As a result, obtain

$$j^n - \frac{3}{[\Gamma(1/3)]^3}\frac{1}{z}j^{n-3} - \sigma\left(\frac{n}{\pi}\sin\frac{\pi}{n}\right)^n = 0. \tag{111}$$

Now make use of the second modification of the algebraic method. According to Table 1 the partial differential equation (52) generates the ordinary differential equation

$$\frac{dj}{dz} = -j^4 + \sigma j^{4-n}. \tag{112}$$

At large z the left-hand side of this equation can be neglected. In this case relationship (109) is preserved between the limiting flow and the parameter σ. At small z the asymptotic solution of equation (112) has the form

$$j_0 = (3z)^{-1/3} \quad (z \to 0). \tag{113}$$

The derivation of this formula was made taking into account condition $j(0) = \infty$.

Expressions (109) and (113) allow equation (112) to be written as

$$j_0^4 \frac{dj}{dj_0} = j^4 - j_\infty^n j^{4-n}. \tag{114}$$

The solution of this equation satisfying the condition $j/j_0 \to 1$ for $j_0 \to \infty$ is given by

$$\zeta = 3\int_y^\infty \frac{dy}{y^4 - y^{4-n}} \tag{115}$$

where the variables y and ζ are related to diffusion flows as

$$y = \frac{j}{j_\infty}, \quad \zeta = \frac{j_\infty^3}{j_0^3}. \tag{116}$$

As usual, the quantities j_0 and j_∞ should be replaced by exact asymptotics which are prescribed by formulae (65) and (32).

To make further calculations, it is necessary to specify the value of the exponent $n = 3$. In this case the solution of the cubic equation (110) in variables (116) can be written as

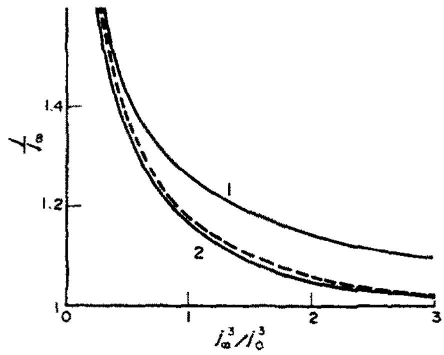

FIG. 3. Diffusion turbulent flow in the tube inlet section obtained by using the first (line 1, equation (117)) and second (line 2, equation (118)) modifications of the algebraic method. The dashed line corresponds to relation (61).

$$y = (1+\zeta^{-1})^{1/3}. \tag{117}$$

Integrating equation (115) at $n = 3$ and making simple transformations give the following relation:

$$y = (1-e^{-\zeta})^{-1/3}. \tag{118}$$

In Fig. 3 solid lines show the results of calculations by formulae (117) and (118) obtained on the basis of the first and second modifications of the algebraic method. The dashed line corresponds to relation (61).

5.3. *Non-stationary heat and mass transfer in turbulent fluid flow far from the tube inlet section*

Analyse the equation

$$\frac{\partial c}{\partial \tau} = \frac{\partial}{\partial x}(1+\sigma x^n)\frac{\partial c}{\partial x} \tag{119}$$

which, together with the initial and boundary conditions (45), describes the model problem of non-stationary heat and mass transfer in turbulent fluid flow at a distance from the tube inlet section.

The use of the first modification of the algebraic method for this problem leads to the equation

$$\frac{1}{\tau} = j^2 - \sigma j^{2-n} \tag{120}$$

the asymptotic solutions of which at small and large times have the form

$$j_0 = 1/\sqrt{\tau} \quad (\tau \to 0). \tag{121}$$

$$j_\infty = \sigma^{1/n} \quad (\tau \to \infty). \tag{122}$$

Equation (120) can be rewritten, with the aid of expressions (121) and (122), as

$$j^2 - j_\infty^n j^{2-n} - j_0^2 = 0. \tag{123}$$

Substituting into equation (123) the exact asymptotic solution of problem (119) and (45), i.e. j_0 and j_∞, which are given by formulae (100) and (32), gives the sought equation

$$j^2 - \sigma\left(\frac{n}{\pi}\sin\frac{\pi}{n}\right)^n j^{2-n} - \frac{1}{\pi\tau} = 0. \tag{124}$$

The second modification of the algebraic method leads to the differential equation

$$-\frac{1}{j}\frac{\mathrm{d}j}{\mathrm{d}\tau}=j^2-\sigma j^{2-n}. \quad (125)$$

The unbounded solution of equation (125) for $\tau \to 0$ is determined by the asymptotic

$$j_0=(2\tau)^{-1/2} \quad (\tau\to 0) \quad (126)$$

whereas for $\tau \to \infty$ equality (122) is valid.

Equation (125) can be presented, with the aid of expressions (122) and (126), in the form

$$j_0^3\frac{\mathrm{d}j}{\mathrm{d}j_0}=j^3-j_\infty^n j^{3-n}. \quad (127)$$

The solution of equation (127) with the property $j/j_0 \to 1$ for $j_0 \to \infty$ is determined as

$$\frac{j_\infty^2}{j_0^2}=2\int_y^\infty \frac{\mathrm{d}y}{y^3-y^{3-n}}, \quad \text{where} \quad y=\frac{j}{j_\infty}. \quad (128)$$

Hence, at $n = 3$ the integration will give

$$\frac{j_\infty^2}{j_0^2}=\frac{1}{3}\ln\frac{j^2+jj_\infty+j_\infty^2}{(j-j_\infty)^2}+\frac{2}{\sqrt{3}}\arctan\left(\frac{2j+j_\infty}{j_\infty\sqrt{3}}\right)-\frac{\pi}{\sqrt{3}}. \quad (129)$$

The exact asymptotics j_0 and j_∞ calculated by formulae (100) and (32) should be substituted into expressions (128) and (129).

5.4. *Non-stationary diffusion to a rotating disk*

Now consider the equation

$$\frac{\partial c}{\partial \tau}-Px^m\frac{\partial c}{\partial x}=\frac{\partial^2 c}{\partial x^2} \quad (130)$$

with initial and boundary conditions (45).

At $m = 1$ and 2, problem (130) and (45) describes a non-stationary concentration field in the vicinity of the forward stagnation point of a bubble and a solid particle in a translational Stokes flow at large Peclet numbers (the parameter P differs from the Peclet number by a numerical factor). To the value $n = 2$ in equation (130) there corresponds a non-stationary diffusion to a flat disk rotating with constant angular velocity in a viscous incompressible fluid [8] (in this case $\tau = Dt/a^2$ and the rest of the dimensionless quantities are introduced in the same manner as in equation (34)).

Further, without any loss of generality, it will be regarded that the parameters m and P in equation (130) can take any non-negative values.

The first modification of the algebraic method (see Table 1) leads to the equation

$$\frac{1}{\tau}+Pj^{1-m}=j^2 \quad (131)$$

the asymptotic solutions of which at small and large τ are determined by the formulae

$$j_0=\tau^{-1/2} \quad (\tau\to 0) \quad (132)$$

$$j_\infty=P^{1/(m+1)} \quad (\tau\to\infty). \quad (133)$$

Eliminating the parameters τ and P from relations (131)–(133) yields

$$j^2-j_\infty^{1+m}j^{1-m}-j_0^2=0. \quad (134)$$

Substituting into this equation the exact asymptotics j_0 and j_∞ which are obtained when studying problem (130) and (45) and which are given by expressions (100) and (41), find the sought algebraic equation for a diffusion flow

$$j^2-(m+1)^m\left[\Gamma\left(\frac{1}{m+1}\right)\right]^{-m-1}Pj^{1-m}-\frac{1}{\pi\tau}=0. \quad (135)$$

In the specific cases of $m = 0$ and 1 the solutions of the quadratic equation (135) have the form

$$j=\frac{P}{2}+\left(\frac{P^2}{4}+\frac{1}{\pi\tau}\right)^{1/2} \quad (\text{at } m=0) \quad (136)$$

$$j=\left(\frac{2P}{\pi}+\frac{1}{\pi\tau}\right)^{1/2} \quad (\text{at } m=1) \quad (137)$$

and the value $m = 2$ corresponds to the cubic equation

$$j^3-(\pi\tau)^{-1/2}j-9[\Gamma(1/3)]^{-3}P=0 \quad (\text{at } m=2). \quad (138)$$

The second modification of the algebraic method in the case considered gives the ordinary differential equation

$$-\frac{1}{j}\frac{\mathrm{d}j}{\mathrm{d}\tau}+Pj^{1-m}=j^2 \quad (139)$$

the asymptotic solutions of which at small and large times are prescribed by formulae (126) and (133), respectively. With the above being taken into account, represent equation (139) in the following form:

$$j_0^3\frac{\mathrm{d}j}{\mathrm{d}j_0}=j^3-j_\infty^{m+1}j^{2-m}. \quad (140)$$

The solution of equation (140) is determined by formula (128) where it should be assumed that $n = m+1$. After this, the exact asymptotics j_0 and j_∞, which are given by expressions (100) and (41), should be substituted. As a result, the procedure described and integration will give

$$\tau=-\frac{2}{\pi P^2}\left[\frac{P}{j}+\ln\left(1-\frac{P}{j}\right)\right] \quad (\text{at } m=0) \quad (141)$$

$$j=\left(\frac{2P}{\pi}\right)^{1/2}[1-\exp(-2P\tau)]^{-1/2} \quad (\text{at } m=1). \quad (142)$$

Here the relation $j = j(\tau)$ at $m = 0$ is written in an implicit form. At $m = 2$, the solution of the equation is given by equality (129), where j_0 and j_∞ are calculated from formulae (100) and (41).

It is important to note that at $m = 1$ formula (142) is exact (this follows from the results of ref. [2]). This very pleasant fact has a much more general character. Namely, it can be shown that the second modification of the algebraic method always leads to an accurate result when the solution of a non-stationary problem is self-similar and can be presented in the form $c = c(x/\delta)$, where $\delta = \delta(\tau)$ is some function of time.

Now, an approximate solution of problem (130) and (45) will be constructed by another technique. For this purpose, the Laplace–Carson transform (46) will be used. As a result, the concentration image $\bar{c}$ will be given by

$$\frac{d^2\bar{c}}{dx^2} + Px^m \frac{d\bar{c}}{dx} - p\bar{c} = 0 \qquad (143)$$

with boundary conditions (48).

With the use of the algebraic method, the equation for the diffusion flow image $\bar{j}$ is derived as

$$\bar{j}^2 - P\bar{j}^{1-m} - p = 0 \qquad (144)$$

the asymptotic solutions of which are

$$\bar{j}_0 = P^{1/(m+1)}(|p| \to 0);\quad \bar{j}_\infty = \sqrt{p}(|p| \to \infty). \qquad (145)$$

Eliminating the parameters P and p from equations (144) and (145) gives

$$\bar{j}^2 - \bar{j}_0^{1+m}\bar{j}^{1-m} - \bar{j}_\infty = 0. \qquad (146)$$

It can be easily shown that the exact solution of problem (143) and (48) leads to the asymptotics

$$\bar{j}_0 = (m+1)^{m/(m+1)}\left[\Gamma\left(\frac{1}{m+1}\right)\right]^{-1} P^{1/(m+1)},\qquad \bar{j}_\infty = \sqrt{p}. \qquad (147)$$

Substitution into equation (146) gives

$$\bar{j}^2 - (m+1)^m \left[\Gamma\left(\frac{1}{m+1}\right)\right]^{-m-1} P\bar{j}^{1-m} - p = 0. \qquad (148)$$

Solution of equation (148) at $m = 0$ and 1 and application of the reverse Laplace–Carson transform yield the following expressions for a diffusion flow:

$$j = \frac{P}{2} + \frac{P}{2}\operatorname{erf}\left(\frac{P}{2}\sqrt{\tau}\right) + \frac{1}{\sqrt{(\pi\tau)}}\exp\left(-\frac{P^2}{4}\tau\right) \quad (\text{at } m = 0) \qquad (149)$$

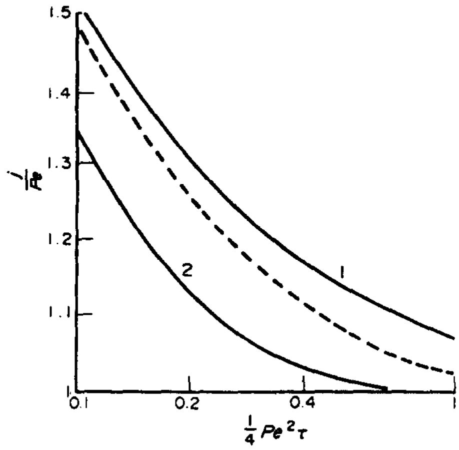

FIG. 4. Comparison of the exact (dashed line, equation (149)) and approximate (solid lines 1 and 2, equations (136) and (141)) expressions for diffusion flows corresponding to the solution of problem (130) and (45) at $m = 0$.

$$j = \left(\frac{2P}{\pi}\right)^{1/2}\operatorname{erf}\left(\frac{2P}{\pi}\tau\right)^{1/2} + \frac{1}{\sqrt{(\pi\tau)}}\exp\left(-\frac{2P}{\pi}\tau\right) \quad (\text{at } m -= 1). \qquad (150)$$

It is important to note that formula (149), obtained by an approximate method is exact.

Figure 4 presents a comparison of the exact, (149) (a dashed line), and approximate, (136) and (141), expressions for a diffusion flow which corresponds to the solution of problem (130) and (45) at $m = 0$. It is seen that the approximate formula (136) obtained with the use of the first modification of the algebraic method gives overestimated values as compared with the exact formula (149), the maximum error in this case is 5.5%. The difference of the approximate expression (141), which is a consequence of the use of the second modification of the algebraic method, from the exact one is around 10%.

Figure 5 shows the dependence of the diffusion flow on time which corresponds to the approximate formula (137) (solid line 1) derived at $m = 1$ by the first modification of the algebraic method; also given is curve (142) (solid line 2) obtained with the use of the second modification of the method. As was noted above, in this case expression (142) coincides with the exact solution. It is seen that the maximum difference of the approximate formula (137) (due to which there are overestimated values) is rather great and amounts to about 14%. The dashed line corresponds to the approximate relation (150) the maximum error of which is equal to 4%.

5.5. Convective mass transfer to a flat plate with a simultaneous volumetric reaction

Consider stationary convective diffusion to the flat plate surface in the translational viscous incompressible fluid flow at large Reynolds numbers (Blau-

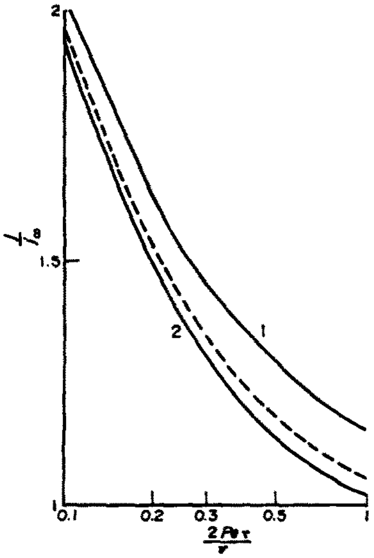

FIG. 5. Relationship between a diffusion flow and time obtained by solving problem (130) and (45) at $m = 1$. Line 1 corresponds to equation (137). Line 2 (accurate result) corresponds to formula (142), the dashed line corresponds to formula (150).

sius flow). It is assumed that mass transfer is compounded with a volumetric reaction. In the diffuse boundary layer approximation a corresponding problem on concentration is described by the equation and boundary conditions

$$\frac{1.33}{4}\frac{x}{z^{1/2}}\frac{\partial c}{\partial z}+\frac{1.33}{16}\frac{x^2}{z^{3/2}}\frac{\partial c}{\partial x}=\frac{\partial^2 c}{\partial x^2}-kf(c) \qquad (151)$$

$$z=0,\ c=0;\quad x=0,\ c=1;\quad x\to\infty,\ c\to 0. \qquad (152)$$

Here, the dimensionless quantities are introduced by the formulae

$$z=\frac{Z}{a},\quad x=\frac{X}{a},\quad c=\frac{C}{C_s},\quad a=\frac{D^{2/3}\nu^{1/3}}{U_\infty},$$

$$k=\frac{a^2KF(C_s)}{DC_s},\quad f(c)=\frac{F(C)}{F(C_s)}$$

where U_∞ is the unperturbed velocity far from the plate, X the distance to the plate surface, Z the distance from the leading edge along a plate.

For convenience, equations (151) and (152) will be replaced by the new variable

$$y=xz^{-1/4}. \qquad (153)$$

This yields a simpler equation

$$\frac{1.33}{4}\frac{y}{z^{1/4}}\frac{\partial c}{\partial z}=\frac{1}{z^{1/2}}\frac{\partial^2 c}{\partial y^2}-kf(c) \qquad (154)$$

with boundary conditions

$$x=0,\ c=0;\quad y=0,\ c=1;\quad y\to\infty,\ c\to 0.$$

In the parabolic equation (154) the role of the time variable is played by z. The unknown quantities $j=-(\partial c/\partial x)_{x=0}$ and $J=-(\partial c/\partial y)_{y=0}$, which correspond to problem (151), (152) and (154), (53), are related as

$$J=z^{1/4}j. \qquad (155)$$

With the use of the first modification of the algebraic method (in Table 1 x, τ, j should be substituted by y, z, J) expression (154) yields

$$\frac{1}{Jz^{5/4}}=\frac{1}{z^{1/2}}J^2-k$$

which, by virtue of equation (155), gives the equation for a diffusion flow

$$j^3-kj-z^{-3/2}=0. \qquad (156)$$

Taking into account the fact that the asymptotic solutions of this equation are determined by

$$j_0=z^{-1/2}\ (z\to 0);\quad j_\infty=k^{1/2}\ (z\to\infty) \qquad (157)$$

equation (156) will be presented in the following form:

$$j^3-j_\infty^2 j-j_0^3=0. \qquad (158)$$

In accordance with the ideas of the algebraic method, not equation (157) but exact asymptotic solutions of the initial problem (151) and (152) for $z\to 0$ [7] and $z\to\infty$ should be substituted into equation (158)

$$j_0=0.399z^{-1/2},\quad j_\infty=(2k\langle f\rangle)^{1/2}. \qquad (159)$$

The above procedure gives

$$j^3-2k\langle f\rangle j-(0.399)^3z^{-3/2}=0. \qquad (160)$$

Note that for the first-order reaction ($f=c$, $f=1/2$) an analogous equation was derived by a different technique in ref. [4].

According to the second modification of the algebraic method, equation (154) generates the ordinary differential equation

$$-\frac{1}{z^{1/4}J^2}\frac{dJ}{dz}=\frac{1}{z^{1/2}}J^2-k \qquad (161)$$

which, with regard for equation (155), can be rewritten as

$$-\frac{1}{z^{3/4}j^2}\frac{d}{dz}(z^{1/4}j)=j^2-k. \qquad (162)$$

The asymptotic solutions of this equation have the form

$$j_0=2^{-2/3}z^{-1/2}\ (z\to 0);\quad j_\infty=k^{1/2}\ (z\to\infty). \qquad (163)$$

Eliminating z and k from expression (162) with the aid of (163) gives

$$\frac{dj}{dj_0}=\frac{1}{2}\frac{j}{j_0}+\frac{j^2(j^2-j_\infty^2)}{j_0^4}. \qquad (164)$$

The sought equation for a diffusion flow is obtained by substituting into equation (164) the exact asymptotics (159)

$$\frac{\mathrm{d}j}{\mathrm{d}z} = -\frac{j}{4z} + \frac{j^2(2k\langle f\rangle - j^2)\sqrt{z}}{2(0.399)^3}. \quad (165)$$

5.6. *Mass transfer in turbulent flow in the tube inlet section with a simultaneous volumetric reaction*

Analyse the following equation:

$$x\frac{\partial c}{\partial z} = \frac{\partial}{\partial x}(1+\sigma x^n)\frac{\partial c}{\partial x} - kf(c) \quad (166)$$

which is a natural generalization of equations (23) and (52). In combination with boundary conditions (53), it describes mass transfer in turbulent flow in the tube inlet section with a simultaneous volumetric reaction.

The first modification of the algebraic method leads to the equation

$$\frac{1}{zj} = j^2 - \sigma j^{2-n} - k \quad (167)$$

the coefficients of which should be refined by the asymptotic correction method.

At a distance from the inlet section for $z \to \infty$ equation (168) changes over equation (26) which, after refinement of the coefficients (by studying the limiting cases $k \to 0$ and $k \to \infty$), is reduced to the form of equation (33). This makes it possible, instead of equation (167), to write directly

$$\frac{1}{zj} = j^2 - \sigma\left(\frac{n}{\pi}\sin\frac{\pi}{n}\right)^n j^{2-n} - 2k\langle f\rangle. \quad (168)$$

At small z the asymptotic of equation (168) is determined by formula (108) thus allowing equation (168) to be presented in the form

$$j^3 - \sigma\left(\frac{n}{\pi}\sin\frac{\pi}{n}\right)^n j^{3-n} - 2k\langle f\rangle j - j_0^3 = 0. \quad (169)$$

Finally, taking into account the fact that the exact asymptotic solution of problem (166) and (52) leads to expression (65), obtain the equation

$$j^3 - \sigma\left(\frac{n}{\pi}\sin\frac{\pi}{n}\right)^n j^{3-n} - 2k\langle f\rangle j - \frac{3}{[\Gamma(1/3)]^3}\frac{1}{z} = 0 \quad (170)$$

which in the limiting cases $z = \infty$ and $k = 0$ changes over to equations (33) and (111), respectively.

According to rule (88), the second modification of the algebraic method instead of equation (168) gives the following ordinary differential equation:

$$\frac{\mathrm{d}j}{\mathrm{d}z} = -j^4 + \sigma\left(\frac{n}{\pi}\sin\frac{\pi}{n}\right)^n j^{4-n} + 2k\langle f\rangle j^2 \quad (171)$$

which changes over to equation (33) for $z \to \infty$.

When $z \to 0$, the asymptotic solution of equation (171) has the form of equation (113).

Eliminating the coordinate z from equation (171) with the aid of equation (113) gives

$$j_0^4\frac{\mathrm{d}j}{\mathrm{d}j_0} = j^4 - \sigma\left(\frac{n}{\pi}\sin\frac{\pi}{n}\right)^n j^{4-n} - 2k\langle f\rangle j^2. \quad (172)$$

Substituting the corresponding exact asymptotic (65) into equation (172), the following equation can be derived:

$$\frac{9}{[\Gamma(1/3)]^3}\frac{\mathrm{d}j}{\mathrm{d}z} = -j^4 + \sigma\left(\frac{n}{\pi}\sin\frac{\pi}{n}\right)^n j^{4-n} + 2k\langle f\rangle j^2. \quad (173)$$

Remark 3. When constructing an approximate equation for a diffusion flow (by any modification of the algebraic method) it is convenient to use the following technique. Each term of the initial partial differential equation is replaced by the quantity taken from the right-hand column of Table 1 and multiplied by some constant. These unknown constants, which will occur in the resulting approximate equation, are then calculated on the basis of an additional asymptotic study of the initial non-stationary problem in some limiting cases. (A similar technique was used in ref. [4] where approximate equations for diffusion flows were derived intuitively.)

For example, acting in such a manner according to the first modification of the method, the following equation will be obtained in the latter problem instead of equation (167):

$$E\frac{1}{zj} = j^2 - A\sigma j^{2-n} - Bk \quad (174)$$

where A, B, E are unknown constants.

Assume now simultaneously in equation (174) that $k = 0$ and $z = \infty$. This will give

$$j = A^{1/n}\sigma^{1/n}. \quad (175)$$

The corresponding exact asymptotic solution of problem (166) and (53) is given by formula (32). Equating expressions (175) and (32) determines the constant A

$$A = \left(\frac{n}{\pi}\sin\frac{\pi}{n}\right)^n. \quad (176)$$

For $z = \infty$ and $k \to \infty$ it follows from equation (174) that

$$j = B^{1/2}k^{1/2}. \quad (177)$$

Comparison of the quantity with the corresponding exact asymptotic (42) yields for the coefficient B

$$B = 2k\langle f\rangle. \quad (178)$$

When $z \to 0$, equation (174) yields

$$j = E^{1/3}z^{-1/3}. \quad (179)$$

Taking into account the fact that in this limiting case the exact asymptotic is given by formula (65), it is possible to find the constant

$$E = 3[\Gamma(1/3)]^{-3}. \qquad (180)$$

Substituting constants (176), (178) and (180) into equation (174), in the long run yields equation (170).

Acting in a similar fashion, the second modification of the algebraic method will give, instead of equation (174), the following equation:

$$-H\frac{1}{j^2}\frac{\mathrm{d}j}{\mathrm{d}z} = j^2 - A\sigma j^{2-n} - Bk. \qquad (181)$$

As before, the unknown constants A and B are found from the analysis of the limiting cases $z = \infty$, $k = 0$ and $z = \infty$, $k \to \infty$ and are determined by formulae (176) and (178). When $z \to 0$, it follows from equation (181) that

$$j = H^{1/3}(3z)^{-1/3}. \qquad (182)$$

An exact asymptotic corresponding to the solution of problem (166) and (53) for $z \to 0$ is given by expression (65). Equating expressions (182) and (65) yields the constant

$$H = 9[\Gamma(1/3)]^{-3}. \qquad (183)$$

Now, by substituting constants (176), (178) and (183) into equation (181) one obtains equation (173).

REFERENCES

1. A. D. Polyanin and V. V. Dil'man, New approximate analytic methods of investigating problems of physicochemical mechanics, *J. Engng Phys.* **46**(3), 415–424 (1984).
2. A. D. Polyanin and V. V. Dil'man, New methods of the mass and heat transfer theory—1. The method of asymptotic correction and the method of model equations and analogies, *Int. J. Heat Mass Transfer* **28**(1), 25–43 (1985).
3. E. Reckenstein, Chemical reaction influence on mass transfer in turbulent fluid flow inside a tube, *Zh. Prikl. Khim.* **38**(6), 1421–1424 (1965).
4. E. Reckenstein, Prediction of rates of heat or mass transfer in complex situations by interpolating between simpler limiting cases, *Chem. Engng Sci.* **37**(10), 1505–1511 (1982).
5. V. A. Ditkin and A. P. Prudnikov, *Handbook on Operation Calculus*. Izd. Vysshaya Shkola, Moscow (1965).
6. V. V. Dil'man and A. D. Polyanin, *Methods of Model Equations and Analogies in Chemical Engineering Science*. Izd. Khimiya, Moscow (1988).
7. V. G. Levich, *Physicochemical Hydrodynamics*. Prentice-Hall, Englewood Cliffs, New Jersey (1962).
8. Yu. P. Gupalo, A. D. Polyanin and Yu. S. Ryazantsev, *Mass and Heat Exchange Between Reacting Particles and a Flow*. Izd. Nauka, Moscow (1985).

METHODE ALGEBRIQUE POUR LES PROBLEMES DE TRANSFERT DE CHALEUR ET DE MASSE

Résumé—On présente une nouvelle méthode analytique approchée pour traiter les problèmes de transfert de chaleur et de masse et obtenir des équations algébriques simples pour les écoulements diffusionnels (thermiques) et des nombres de Sherwood moyens. Les problèmes considérés concernent le transfert de masse pour un disque tournant, une plaque dans le sens de l'écoulement, une gouttelette sphérique et une particule solide au cours d'une réaction volumétrique dont la vitesse est fonction de la concentration. On étudie l'echange variable de masse entre une paroi et le milieu au repos, pour une réaction simultanée d'ordre quelconque. Sont analysés le transfert de chaleur et de masse dans un écoulement turbulent à l'entrée du tube, et d'autres problèmes analogues.

EINE ALGEBRAISCHE METHODE ZUR LÖSUNG VON WÄRME- UND STOFFÜBERTRAGUNGSPROBLEMEN

Zusammenfassung—Es wird eine neue analytische Näherungsmethode zur Untersuchung von Wärme- und Stoffübertragungsproblemen vorgestellt, die es ermöglicht, thermische Diffusionsströmungen und mittlere Sherwood-Zahlen mit einfachen algebraischen Gleichungen zu beschreiben. Es können Probleme gelöst werden, die beim konvektiven Stoffübergang an einer rotierenden Scheibe auftreten, an einer längs angeströmten Platte, an einem kugelförmigen Tröpfchen und an einem festen Teilchen in Abhängigkeit einer volumetrischen Reaktion, deren Intensität beliebig konzentrationsabhängig sein kann. Es wird der instationäre Stoffaustausch zwischen einer Wand und einem ruhenden Fluid bei einer gleichzeitig auftretenden volumetrischen Reaktion beliebiger Ordnung untersucht. Es wird der Wärme- und Stoffaustausch im Einlaßbereich eines turbulent durchströmten Rohres sowie ähnlicher Probleme untersucht.

АЛГЕБРАИЧЕСКИЙ МЕТОД ИССЛЕДОВАНИЯ ЗАДАЧ МАССО- И ТЕПЛОПЕРЕНОСА

Аннотация—Излагается новый приближенный аналитический метод исследования задач массо- и теплопереноса, позволяющий получать простые алгебраические уравнения для диффузионных (тепловых) потоков и средних чисел Шервуда. Рассмотрены задачи о конвективном массопереносе к вращающемуся диску, продольно обтекаемой пластинке, сферической капле и твердой частице, при протекании объемной реакции, скорость которой произвольным образом зависит от концентрации. Исследован нестационарный массообмен стенки с неподвижной средой, осложненный объемной реакцией любого порядка. Анализируются тепло- и массоперенос при турбулентном течении на входном участке трубы и другие задачи такого типа.